Thermodynamic Properties of a Solid Exhibiting the Energy Spectrum given by the Logistic Map

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Abstract

We show that the infinite-dimensional representation of the recently introduced Logistic algebra can be interpreted as a non-trivial generalization of the Heisenberg or oscillator algebra. This allow us to construct a quantum Hamiltonian having the energy spectrum given by the logistic map. We analyze the Hamiltonian of a solid whose collective modes of vibration are described by this generalized oscillator and compute the thermodynamic properties of the model in the two-cycle and $r = 3.6785$ chaotic region of the logistic map.

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1 Introduction

In the last years, complex systems have attracted a lot of attention. In particular, there is an intrinsic theoretical interest in constructing a Hamiltonian system having an energy spectrum that is or quasi-periodic or self-similar, and/or chaotic [1]. Enhancing the interest in describing such Hamiltonian system is the fact that some models on quasicrystals have a quasi-periodic or fractal energy spectrum [2, 3, 4, 5, 6, 7, 8]. On the other hand, one paradigmatic example of map that exhibits some of these features is the logistic map. As is well-known, this map describes at the Feigenbaum point an example of a fractal system and appearing after this point a chaotic region with chaotic bands and self-similar patterns [9].

Recently, it was developed a three-generator algebra, called Logistic algebra [10], where the eigenvalue of one generator is given by the logistic map . We show that the infinite-dimensional representation of this algebra can be interpreted as a non-trivial generalization of the Heisenberg or oscillator algebra and call the associated oscillators, logistic oscillators.

We use these logistic oscillators to construct a quantum Hamiltonian, which is a generalization of the quantum harmonic oscillator, that has the energy spectrum described by the logistic map. We apply these ideas to construct a Hamiltonian describing quasi-particle vibrations of a solid with N atoms where each quasi-particle oscillates as a logistic oscillator.

In section 2, we discuss the logistic algebra and its interpretation as generalized Heisenberg algebra, in section 3 we construct a model for a solid where the collective modes of motion are described as logistic oscillators and compute the thermodynamic functions of the model in the two-cycle and $r = 3.6785$ chaotic region of the logistic map $x_{n+1} = rx_n(1 - x_n)$. Section IV is devoted to our conclusions.

2 Algebraic origin of the model

The model we are going to discuss in sections 3 and 4 has its origin in an algebraic structure called Logistic algebra [10]. In this section we present the Logistic algebra and show that this algebra can be interpreted as a non-trivial extension of Heisenberg algebra.

Let us consider the algebra generated by J_0 , $J_\pm$, described by the relations [10]

$$J_i J_+ = J_+ J_{i+1} \quad i = 0, 1, 2, \ldots \quad , \tag{1}$$

$$J_- J_i = J_{i+1} J_- \quad , \tag{2}$$

$$J_+ J_- - J_- J_+ = -a(J_0 - J_1) \quad , \tag{3}$$

where $J_- = J_+^\dagger$, $J_i^\dagger = J_i$ and "a" is a real constant. Moreover,

$$J_{i+1} = r J_i (1 - J_i) \quad i = 0, 1, 2, \ldots \quad , \tag{4}$$

with $0 \leq r \leq 4$.

The hermitian operator J_0 can be diagonalized. Consider the state $|0>$ with the lowest [1] eigenvalue of J_0

$$J_0 |0> = \alpha_0 |0> \quad . \tag{5}$$

Note that, for each value of α_0 we have a different vacuum that for simplicity all of them will be denoted by $|0>$. We choose $0 \leq \alpha_0 \leq 1$ because with this condition all the future iteration will remain in this interval and the connection with the chaotic concepts is straightforward. Also, the allowed values of α_0 depend on r and a. Since, by hypothesis, α_0 is the lowest J_0 eigenvalue we must have,

$$J_- |0> = 0 \quad . \tag{6}$$

Following the usual steps for constructing (now from lower to higher eigenvalues) SU(2) algebra representations [11], using the algebraic relations exhibited in eqs. (1,2,3) and taking into account eqs. (5,6) we obtain:

$$J_0 |m> = \alpha_m |m> \quad , \tag{7}$$

$$J_+ |m> = N_m |m+1> \quad , \tag{8}$$

$$J_- |m+1> = N_m |m> \quad , \tag{9}$$

where [2]

[1]Due to the use of the logistic map, depending on the values of r and α_0 considered, $|0>$ can be the state with highest weight. We emphasize in this paper the case where $|0>$ is a lowest weight vector since it is the situation considered in the following sections.

[2]Note that if we put $m = -1$ in the eq. (9) we obtain consistently eq. (6).

$$N_m = \sqrt{a(\alpha_0 - \alpha_{m+1})} \quad , \tag{10}$$

and

$$\alpha_{m+1} = r\alpha_m(1 - \alpha_m) \quad . \tag{11}$$

Note that the states $|m>$, $m \geq 1$, are defined by the application of J_+ on $|m-1>$. Moreover, from eqs. (7 - 9) we easily obtain a general expression for $|m>$,

$$|m> = \frac{1}{\prod_{i=0}^{m-1} N_i}(J_+)^m|0> \quad . \tag{12}$$

Of course, since the eigenvalues of J_0 are given by the logistic map (eq. (11)), their values as m increases can have an irregular behavior depending on the values of r and α_0 , and the dimension of the representation. Note that, unlike $SU(2)$ algebra where the states obtained by the application of J_+ always have higher J_0 eigenvalues, for the logistic algebra this depends on what values of r and α_0 we consider and the level of iterations (the number m of $|m>$) we are. For instance, for $r = 3$ and $\alpha_m = 0.5$ we have $\alpha_{m+1} = 0.75$, i.e., J_+ rises the J_0 eigenvalue of $|m>$. On the other hand, for $r = 1.5$ and $\alpha_m = 0.5$ we have $\alpha_{m+1} = 0.375$ and in this case J_+ lowers the J_0 eigenvalue of $|m>$. Moreover, due to the non-regular behavior of the logistic map, it may happen for J_+ that even having started as lowering the J_0 eigenvalue of $|m>$ it rises the J_0 eigenvalue of $J_+|m>$ for a given level m of iteration of the logistic map. For instance, for $r = 2.75$ and $\alpha_m = 0.9$ we have $\alpha_{m+1} = 0.247$ and $\alpha_{m+2} = 0.5122$.

Let us now consider the operator

$$C = J_+J_- + aJ_0 = J_-J_+ + aJ_1 \quad . \tag{13}$$

Using the algebraic relations (eqs. (1,2,3)) it is easy to see that

$$[C, J_0] = [C, J_\pm] = 0 \quad , \tag{14}$$

i.e., C is the Casimir operator of the algebra. In fact, we arrive easily at

$$C|m> = c_0|m> \quad , \tag{15}$$

with $c_0 = a\alpha_0$ independent of m .

With respect to matrix representations of the Logistic algebra there are finite-dimensional matrix representations corresponding to the n-cycle solutions of the logistic map and infinite-dimensional ones relative to the n-cycle and to the chaotic regime of the logistic map. Here we present some examples:

1. Two-Dimensional Representations

$$J_0 = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_1 \end{pmatrix} \quad , \quad J_+ = \begin{pmatrix} 0 & 0 \\ N_0 & 0 \end{pmatrix} \quad , \quad J_- = J_+^\dagger \quad . \tag{16}$$

The allowed values of r and α_0 are determined by the equation $N_1^2 = 0$ such that $N_0^2 \neq 0$. There are two non-trivial solutions,

$$\alpha_0^\pm = \frac{r + 1 \pm \sqrt{r^2 - 2r - 3}}{2r} \tag{17}$$

The solution α_0^+ gives $\alpha_0^+ > \alpha_1^+$ implying $a > 0$, while $\alpha_0^- < \alpha_1^-$ gives $a < 0$. For both cases $r \geq 3$. We will use this solution in the next section.

2. Three-Dimensional Representations

$$J_0 = \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix} \quad , \quad J_+ = \begin{pmatrix} 0 & 0 & 0 \\ N_0 & 0 & 0 \\ 0 & N_1 & 0 \end{pmatrix} \quad , \quad J_- = J_+^\dagger \ . \tag{18}$$

The allowed values of r and α_0 are computed from $N_2 = 0$, $N_0, N_1 \neq 0$.

3. Infinite-Dimensional Representations

$$J_0 = \begin{pmatrix} \alpha_0 & 0 & 0 & 0 & \dots \\ 0 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & \alpha_2 & 0 & \dots \\ 0 & 0 & 0 & \alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ J_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ N_0 & 0 & 0 & 0 & \dots \\ 0 & N_1 & 0 & 0 & \dots \\ 0 & 0 & N_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ J_- = J_+^\dagger \ . \tag{19}$$

The allowed values of r and α_0 can be computed for instance for $a < 0$ from $N_m^2 = |a|(\alpha_{m+1} - \alpha_0)$ by imposing $\alpha_m > \alpha_0$ for all values of $m \geq 1$. In fig. 1 we show a half-leaf region with the allowed values of r and α_0 satisfying the above requirements. These solutions will be used in the following section.

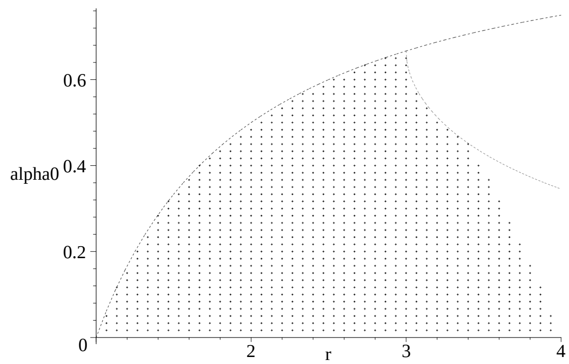

Figure 1: Region of allowed values for α_0 and r.

Let us show now an interesting connection of this algebra with the Heisenberg algebra. The Heisenberg algebra is generated by the elements A and A^+ satisfying the relations

$$AA^\dagger - A^\dagger A = 1 \quad , \tag{20}$$

$$NA^\dagger - A^\dagger N = A^\dagger \quad , \tag{21}$$

with $N = A^\dagger A$ is the number operator. Note that eqs. (1, 2), for $i = 0$ can be seen as defining equations for J_1. The Heisenberg algebra comes naturally if we put in eqs. (1,2, 3) $J_- \equiv A$, $J_+ \equiv A^\dagger$, $J_0 \equiv N$, $J_1 = J_0 + 1$ and $a = -1$. It can be easily verified that we do not have in this case finite dimensional representations and the Casimir operator is identically null.

In summary, Heisenberg algebra is the special case of the defining relations given by eqs. (1,2, 3) where instead of taking the relation given by eq. (4) we consider the simpler one $J_1 = J_0 + 1$. In other words, the Logistic algebra can be interpreted as an extension of the Heisenberg algebra where instead of the simple iteration $J_{i+1} = J_i + 1$ we take the logistic map for J_{i+1} as in eq. (4) . Clearly, it is also possible here to consider other maps; this study is under progress .

Of course, since the Heisenberg algebra is a master algebra in physics, it is a natural step to investigate the possible consequences of the logistic generalization, explained before, in physical problems. In the following sections we apply this generalized Heisenberg algebra to a collective modes of motion of N atoms.

3 Model and thermodynamic properties

Let us consider the Hamiltonian of a quantum system of quasi-particles described by N independent, localized, "oscillators" of the form

$$H = \sum_{q=1}^{N} \epsilon_q J_0^q \quad , \tag{22}$$

where $\{J_0^q\}$ is a collection of N independent oscillators, each of them described by the algebra (1-3), and ϵ_q is a parameter associated to the energy of the q-th oscillator. We are then considering independent collective excitations with a non-trivial spectrum specified by the eigenvalues of J_0^q. For the solution $J_- = A$, $J_+ = A^\dagger$, $J_0 = N$, $J_{i+1} = J_i + 1$ and $a = -1$ of the algebra (1-3), the Hamiltonian (22) describes the well-known system of N independent, localized, harmonic oscillators. On the other hand, by considering the logistic generalization, eq. (22) becomes the Hamiltonian of a system of quasi-particles described by N independent, localized, logistic oscillators. We interpret J_-^q, J_+^q and J_0^q as annihilation, creation and generalized number operator, respectively, of the q-th oscillator. Note that the energy of the q-th oscillation-mode in a state $|m>$ is given by the product of ϵ_q times the eigenvalue of $\{J_0^q\}$ applied on that state. The eigenvalue α_m^q indicates that the q-th oscillation-mode is in the state $|m>$. We are adopting this model due to its simplicity but these logistic oscillators could also be used in more complicated models as for example in disordered systems.

The partition function of the model (22)

$$Z = Tr \exp(-\beta H) \quad , \tag{23}$$

with $\beta = (k_B T)^{-1}$ and k_B the Boltzmann constant, factorizes into a product of single particle partition functions,

$$Z = \prod_q Z_q \quad , \tag{24}$$

$$Z_q = \sum_{m=0}^{\infty} \exp\left(-\beta\epsilon_q \alpha_m^q\right) \quad , \tag{25}$$

where the trace was performed using the basis described in eqs. (5 - 14) and $\alpha_{m+1} = r\alpha_m(1-\alpha_m)$. We take the simplest case where α_0 and r are independent of q.

We suppose that the dispersion relation of the quasi-particle is given by (equivalent to the Debye approximation),

$$\epsilon_q = \epsilon(q) = \gamma q \quad , \tag{26}$$

and we enclose the system in a large 3-dimensional volume V. Replacing, in the usual way (since we are considering phonons with a spectrum different from the harmonic oscillator one), the sum over particles by an integral over a q-space,

$$\sum_q \rightarrow \frac{V}{(2\pi)^3}\int d^3q \quad , \tag{27}$$

we obtain, for the logarithm of the partition function, after integrating over the angular variables,

$$lnZ = \frac{V}{2\pi^2}\int_0^{q_M} dq q^2 \ln\left(\sum_{m=0}^{\infty}\exp(-\beta\gamma q\alpha_m)\right) \quad , \tag{28}$$

where this integral is evaluated over a finite q-range corresponding to a finite number of oscillators, and q_M is the larger possible number q. The mean energy of the solid becomes after defining a new variable $\eta = \beta\gamma q$, $E_0 \equiv \gamma q_M$, $T_0 = E_0/k_B$ and $A \equiv \frac{Vq_M^3}{2\pi^2}$

$$E = -\frac{\partial lnZ}{\partial\beta} = AE_0\left(\frac{T}{T_0}\right)^4\int_0^{T_0/T} d\eta\eta^3\frac{\sum_{m=0}^{\infty}\alpha_m\exp(-\eta\alpha_m)}{\sum_{m=0}^{\infty}\exp(-\eta\alpha_m)} \quad . \tag{29}$$

Let us study the integrand of eq. (29) . The sum is performed over the integer m that corresponds to the level of iteration of the logistic map since α_m is given by this map. In what follows we shall consider two cases: an example of the two-cycle and another one corresponding to the chaotic region of the logistic map.

At a given approximation, in the two-cycle region of the logistic map ($3 < r < 3.449489\ldots$), the iteration runs over transient states before reaching

the asymptotic two levels, which are infinitely degenerated. Clearly, when the degeneracy g of the two levels goes to infinity the contribution of the transient states disappears and only the contribution of the states related to the asymptotic levels remains. The measure is concentrated on the two asymptotic levels. The effective expression for the energy in the infinite g limit is given by:

$$E = AE_0(\frac{T}{T_0})^4 \int_0^{T_0/T} d\eta\eta^3 \frac{\alpha^- \exp(-\eta\alpha^-) + \alpha^+ \exp(-\eta\alpha^+)}{\exp(-\eta\alpha^-) + \exp(-\eta\alpha^+)} \; . \qquad (30)$$

For the specific heat at constant volume we have:

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = Ak_B(\frac{T}{T_0})^3 \left[4\int_0^{T_0/T} d\eta\eta^3 f(\eta) - (\frac{T_0}{T})^4 f(T_0/T)\right] \;, \qquad (31)$$

where

$$f(\eta) = \frac{\alpha^- \exp(-\eta\alpha^-) + \alpha^+ \exp(-\eta\alpha^+)}{\exp(-\eta\alpha^-) + \exp(-\eta\alpha^+)} \quad . \qquad (32)$$

In figure 2 we display $e \equiv E/AE_0$ times $t \equiv T/T_0$; in figure 3 we show $C \equiv C_V/C_0$ times t with $C_0 \equiv Ak_B$. These are typical graphics for two-level systems since after the transient states what remains is the two-cycle situation. For higher-cycle regions of the logistic map we shall have the typical behavior of a system with a finite number of levels.

If we calculate the entropy from eq. (28) we see that it diverges, since the degeneracy factor g goes to infinity. The renormalized entropy $S_R \equiv (S/k - (A/3)\log g)/A$ can be calculated and expressed as:

$$\begin{aligned} S_R &= (\frac{T}{T_0})^3 \left[\int_0^{T_0/T} d\eta\eta^2 \log\Big(\exp(-\eta\alpha^+) + \exp(-\eta\alpha^-)\Big) + \right. \\ &\qquad \left. \int_0^{T_0/T} d\eta\eta^3 f(\eta)\right] \quad . \qquad (33) \end{aligned}$$

More interesting is the behavior of the system we are analysing for the chaotic region. In this case we have as before transient states with the difference that instead of having a finite number of asymptotic levels we have a continuum of levels similar to the classical continuum levels in a classical system. Thus, after dropping the transient states, as the measure is concentrated on the

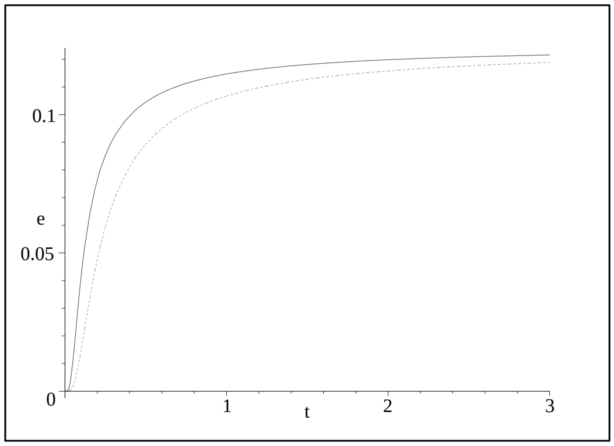

Figure 2: Two cycle energy versus temperature. Continuous line, $r = 3.1$. Broken line, $r = 3.35$

chaotic region, the system is better described by a density function that represents the number of hits of the logistic map in the interval $[0, 1]$. In this case the mean energy is given by

$$E = AE_0(\frac{T}{T_0})^4 \int_0^{T_0/T} d\eta \eta^3 F(\eta) \quad , \tag{34}$$

with

$$F(\eta) = \frac{\int_0^1 d\xi \xi H(\xi) \exp(-\eta\xi)}{\int_0^1 d\xi H(\xi) \exp(-\eta\xi)}, \tag{35}$$

where $H(\xi)$ is the density function. The specific heat at constant volume becomes

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = C_0(\frac{T}{T_0})^3 \left[4 \int_0^{T_0/T} d\eta \eta^3 F(\eta) - (\frac{T_0}{T})^4 F(\frac{T_0}{T})\right] \quad . \tag{36}$$

In figures 4 and 5 we show E/AE_0 and C_V/C_0 times T/T_0 in the chaotic region for $r = 3.6785$.

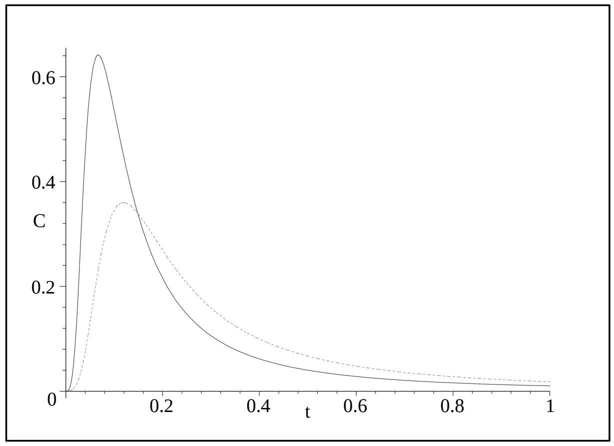

Figure 3: Specific heat for a two cycle. Continuous line, $r = 3.1$; broken line, $r = 3.35$.

Figures 4 and 5 exhibit the typical low-temperature behavior of a classical system as we had already anticipated, since we have a continuum energy level in the chaotic band. As the spectrum is limited from above, the behavior of the specific heat, at high temperatures, for any value of r, is proportional to $1/T^2$ as expected from systems that present the Schottky anomaly.

We also show in fig. 6 the normalized histogram and the density function

$$H(\xi) = \begin{cases} \left((\pi/2)\sqrt{(\xi - 0.266)(0.726 - \xi)} \right)^{-1} & if \quad 0.266 \leq \xi \leq 0.726 \ , \\ \left((\pi/2)\sqrt{(\xi - 0.728)(0.922 - \xi)} \right)^{-1} & if \quad 0.728 \leq \xi \leq 0.922 \ , \\ 0 & otherwise \ , \end{cases} \tag{37}$$

we used in order to compute the mean energy, the specific heat and the entropy.

After dropping the transient states, the entropy for the chaotic region $S_{chaos} \equiv S/S_0$, where $S_0 = C_0 = Ak_B$, can be expressed as:

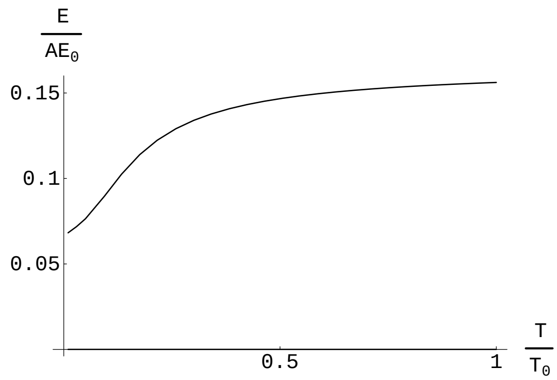

Figure 4: Energy versus temperature for a chaotic spectrum.

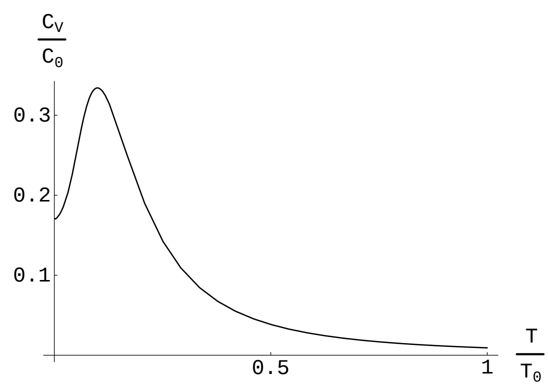

Figure 5: Specific heat versus temperature for the chaotic spectrum.

$$
\begin{aligned}
S_{chaos} &= (\frac{T}{T_0})^3 \left[\int_0^{T_0/T} d\eta \eta^2 \log \left(\int_0^1 d\xi H(\xi) \exp(-\eta\xi) \right) + \right. \\
&\left. \int_0^{T_0/T} d\eta \eta^3 F(\eta) \right] \quad . \qquad (38)
\end{aligned}
$$

The entropy of the chaotic band, showed in fig. 7, presents a curious behavior. In fact, its low-temperature behavior is typical of a classical system, with a negative divergence as $T \rightarrow 0$. On the other hand, its high-temperature behavior is typical of a system with a limited spectrum, found mainly in quantum systems. The origin of this hybrid behavior is the fact that the neighbor levels in the chaotic band have no minimum distance among them,

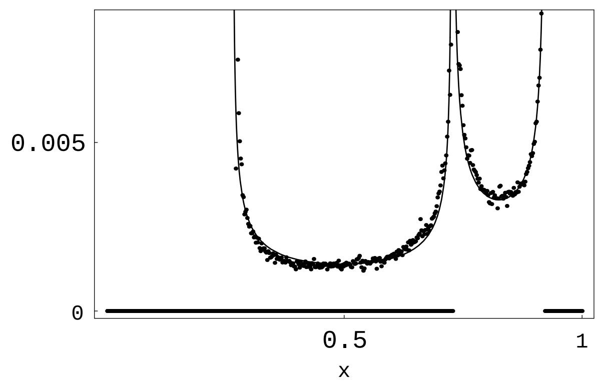

Figure 6: Histogram of the logistic map for $r = 3.6785$ and the function we used (full line) to calculate the thermodynamic functions.

since they are dense inside the chaotic band. This is somewhat equivalent of taking $\hbar \to 0$ limit, thus reobtaining classical low-temperature behaviors. Note that in this case this limit is not imposed, but it is intrinsic of the system since the commutation relations are always different from zero.

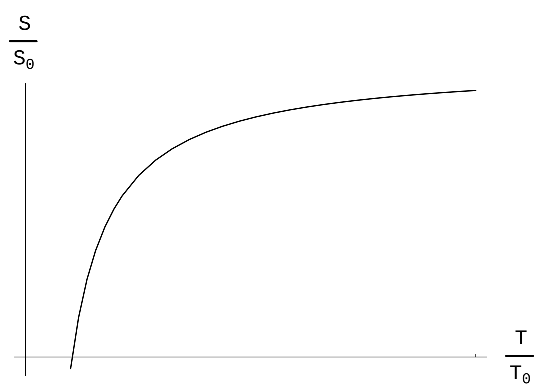

Figure 7: Entropy versus temperature for a chaotic spectrum.

4 Conclusion

We construct, based on an algebra developed in [10], a Hamiltonian of a quasi-particle that presents an energy spectrum whose energy levels are generated by the logistic map. Depending on the parameter r of the logistic map,

the energy levels can be finite (corresponding to cycles of the logistic map) or chaotic (corresponding to the chaotic bands of the map). We study the thermodynamic properties of a Debye-like solid constituted by these elementary quasi-particles and we exhibit the behavior of some thermodynamical quantities like internal energy, specific heat and entropy. These functions, associated to the chaotic spectrum, present a mixed aspect, with both classical and quantum-like typical behaviors. This is consequence of the fact that the spectrum in the chaotic region is continuous, similar to the spectra of classical systems, with no separation between neighboring levels. On the other hand, the thermodynamic quantities related to the cycles are equivalent to systems with a finite number of energy-levels.

It is interesting to note that the algebraic formalism developed in section 2 works consistently for a large class of maps $J_{i+1} = f(J_i)$. Of course, changing eqs. (4, 11) implies a different representation theory of the algebra (1-3) and a different physical Hamiltonian.

A classification of the analytical functions f under a stability theory would lead us to determine the different Hamiltonians associated to the different kinds of attractors of the map f. A systematic study of different non-trivial relations $J_{i+1} = f(J_i)$ and their consequences on the Hamiltonian spectra is under study.

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References

[1] K. Nakamura, "Quantum chaos: a new paradigm of nonlinear dynamics", Cambridge Nonlinear Science Series 3, C.U.P. 1993.

[2] J.B. Sokoloff, Physics Reports, **126**, 189 (1985); B. Simon, Adv. Appl. Math. **3**, 463 (1982); H. Hiramoto and M. Kohomoto, Int. J. Mod. Phys. B **6**, 281 (1992); R. Fleischmann, T. Geisel, R. Ketzmerick and G. Petschel, Physica D **86**, 171 (1995).

[3] C. Tsallis, L.R. da Silva, R.S. Mendes, R.O. Vallejos and A.M. Mariz, Phys. Rev. E, **56**, R4922 (1997); R.O. Vallejos, R.S. Mendes, L.R. da Silva and C. Tsallis, Phys. Rev. E, **58**, 1346 (1998).

[4] R. O. Vallejos and C. Anteneodo, Physical Review E **58**, 4134-4140 (1998).

[5] P.M.C. de Oliveira, E.L. Albuquerque and A.M. Mariz, Physica A,**227**, 206 (1996).

[6] P.E. de Brito, C.A.A. da Silva and H.N. Nazareno, Phys. Rev. B **51**, 6096 (1995); P.E. de Brito, J.A. Gonzalez, H.N. Nazareno, Phys. Rev. B **54**, 12820 (1996); H.N. Nazareno, J.A. Gonzalez and I.F. Costa, Phys. Rev. B **57**, 13583 (1998).

[7] C.S. Ryu, G.Y.Oh and M.H. Lee, Phys. Rev. B, **46**, 5162 (1992).

[8] C. Tsallis, L.R. da Silva and R.O. Vallejos, Private communication.

[9] H.-O. Peitgen, H. Jurgens and D. Saupe, "Chaos and Fractals", Springer-Verlag, New York, 1992.

[10] M. A. Rego-Monteiro, "Logistic Algebras", Preprint cbpf-nf-023/99, (submitted for publication).

[11] H. Georgi, "Lie Algebras in Particle Physics" , Benjamin /Cummings, London, 1982, chapter III, pages 16-19.